INVARIANT SOLUTIONS OF THE EQUATIONS OF MOTION OF IONIZED GAS

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Group analysis of the equations of motion of ionized gas heated by external radiation is performed. The problem of self-similar flow of completely ionized gas is solved numerically.

In considering various problems of the motion of two-phase media and plasma theory, there arises the need to investigate the properties of partial differential equations describing the dynamics of a two-component gas. In the present work, some particular solutions of the equations of motion of ionized gas are found and investigated. To obtain the solutions, the group-theory methods developed in [1] are used. The solutions obtained belong to the class of invariant solutions.

1. The equations of motion of nonviscous and non-heat-conducting ionized gas are written in the form

$$\begin{gathered} F_1=\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0, \\ F_2=\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}=0, \\ F_3=\rho\frac{\partial\varepsilon}{\partial t}+\rho u\frac{\partial\varepsilon}{\partial x}+p\frac{\partial u}{\partial x}-Q=0, \\ F_4=\rho\frac{\partial\varepsilon_i}{\partial t}+\rho u\frac{\partial\varepsilon_i}{\partial x}+p_i\frac{\partial u}{\partial x}+Q_{ei}=0, \\ p_{i,e}=(\gamma-1)\rho\varepsilon_{i,e},\ \varepsilon_{i,e}=c_{i,e}T_{i,e},\ p=p_i+p_e. \end{gathered} \tag{1}$$

The following relations are specified for Q

$$Q=\varkappa q,\ q=q_0t^n\exp\left(-\int_x^\infty\varkappa dx\right),\ q_0,\ n=\text{const}. \tag{2}$$

The exchange energy Q_{ei} may be written in the form [2]

$$Q_{ei}=Q^0_{ei}(\varepsilon_e-A\varepsilon_i),\ A=c_e/c_i. \tag{3}$$

Suppose that the coefficients Q^0_{ei} and $\varkappa$ may be written in power-law form

$$Q^0_{ei}=Q_0\rho^\alpha\varepsilon_e^{\alpha_1},\ \varkappa=\varkappa_0\varepsilon_e^{\alpha_2}\rho^{\alpha_3},\ Q_0=\text{const},\ \varkappa_0=\text{const}. \tag{4}$$

Group analysis of the system in Eqs. (1)-(4) is performed, and invariant solutions of this system are found. The basic group is found by the standard means [1], considering an infinitesimal operator of the space E_6 of variables $(\vec{x},\vec{v})=\{x, t, \rho, u, \varepsilon_i, \varepsilon_e\}$:

$$X=\xi^i(\vec{x},\vec{v})\frac{\partial}{\partial x^i}+\eta^i(\vec{x},\vec{v})\frac{\partial}{\partial v^i}. \tag{5}$$

The continuation of X in the first derivatives $p^j_k=\partial v^j/\partial x^k$ takes the form [1]

$$\tilde{X}=X+\zeta^i_\alpha\frac{\partial}{\partial p^\alpha_i}, \tag{6}$$

where

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$$\zeta_\alpha^j = D_\alpha \eta^j - p_k^j D_\alpha \xi^k;\; D_\alpha = \frac{\partial}{\partial x^\alpha} + p_\alpha^j \frac{\partial}{\partial v^j}.$$

The system in Eq. (1) admits the operator in Eq. (5) if finite transformations of the single-parameter group produced by this operator transform each solution of Eq. (1) back into the solution of this system. In terms of the continuation $\tilde{X}$ of the operator in Eq. (6), this is the condition of invariance of Eq. (1) as a manifold in the space $(\vec{x}, \vec{v})$ and of the derivatives of p_k^j with respect to $\tilde{X}$, i.e., it must be the case that $\tilde{X}F_i = 0$ on the manifolds $F_i = 0$ $(i = 1, 2, 3, 4)$. The latter leads to the existence of the following basis operator groups

$$X_1 = \frac{\partial}{\partial x},\; X_2 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \tag{7}$$

$$X_3 = \alpha_1 t\frac{\partial}{\partial t} + \left(\alpha_1 - \frac{1}{2}\right) x\frac{\partial}{\partial x} - \frac{u}{2}\frac{\partial}{\partial u} - \frac{s}{2}\rho\frac{\partial}{\partial \rho} - \varepsilon_e\frac{\partial}{\partial \varepsilon_e} - \varepsilon_i\frac{\partial}{\partial \varepsilon_i},$$

where $s = 2[1 - \alpha_2 + (n + 1)\alpha_1]/(\alpha_3 - 1)$; $s(\alpha - 1) = 0$.

By the methods of [1], it may be shown that the optimal system of operators (subalgebra) takes the form X_1, X_2, X_3, $X_1 + X_2$, $X_2 + X_3$, $X_1 + X_3$. The corresponding invariant solutions are

$$u = U(t),\; \rho = R(t),\; p = \mathscr{P}(t);\; u = x/t + U(t),\; \rho = R(t),\; p = \mathscr{P}(t);$$

$$u = \frac{x}{t}U(\lambda),\; \rho = \left(\frac{x}{t}\right)^s R(\lambda),\; p = \left(\frac{x}{t}\right)^{s+2}\mathscr{P}(\lambda),$$

$$\lambda = x/t^\delta,\; \delta = (2\alpha_1 - 1)/2\alpha_1;$$

$$u = x/(1 + t) + U(t),\; \rho = R(t),\; p = \mathscr{P}(t);$$

$$u = 2 + U(\lambda)/t^{1/2\alpha_1},\; \rho = R(\lambda)/t^{s/2\alpha_1},$$

$$p = \mathscr{P}(\lambda)/t^{\frac{s+2}{2\alpha_1}},\; \lambda = \left[x + \frac{t}{\alpha_1(\delta - 1)}\right]/t^\delta;\; u = U(\lambda)/t^{1/2\alpha_1},$$

$$\rho = R(\lambda)/t^{s/2\alpha_1},\; p = \mathscr{P}(\lambda)/t^{\frac{s+2}{2\alpha_1}},\; \lambda = (x + t/\alpha_1\delta)/t^\delta.$$

There are no other invariant solutions in nonsimilar subgroups.

Consider the invariant solution corresponding to the operator group $G\langle X^3\rangle$. Since this group is an extension group, the given solution will be self-similar [1]. In [3], it was shown that algebraic integrals may be obtained for self-similar motions.

When $\alpha \neq 1$, the self-similar solutions existing in the extension group $G\langle X_3\rangle$ will hold when $s = 0$, while $\alpha_2 = 1 + (n + 1)\alpha_1 - \alpha$ and α_3 are arbitrary. For a completely ionized plasma [2], $\alpha_1 = -3/2$, $\alpha_2 = -2$, $\alpha = \alpha_3 = 2$, which is obtained with linear energy supply $n = 1$.

2. The self-similar class of flows will now be examined in more detail; consider the following problem. Suppose that, in the cross section $x = 0$, at time $t = 0$, the supply of completely ionized gas is switched on; the gas is modeled by surface mass sources $m = m_0 t^l$ and energy sources $w = w_0 t^k$. It is clear that, if the problem is self-similar (under the assumption of strong perturbations), it is necessary to choose $l = 1/3$, $k = 1$, $n = 1$ [4]. Investigation of the structure of the nonsteady motion arising demonstrates (when $t > 0$, m_0, $w_0 \neq 0$) that, a shock wave will propagate along the surrounding gas; behind the shock wave, right up to the contact discontinuity x_p, there is a region of transparent and equilibrium compressed gas (here $\varkappa_0 = Q_0 = 0$, the flow in this region is described by the corresponding problem of a piston [4]). From the contact surface x_p to the cross section $x = 0$, there exists a continuous solution described by Eq. (1). Self-similar variables are introduced as follows

$$f(\lambda) = u/u_2,\; g(\lambda) = \rho/\rho_2,\; h(\lambda) = p/p_2, \tag{8}$$

$$\theta(\lambda) = \varepsilon/\varepsilon_2,\; \lambda = x/x_2,\; x_2 = Q_0^{1/3} t^\delta,\; \delta = 4/3,$$

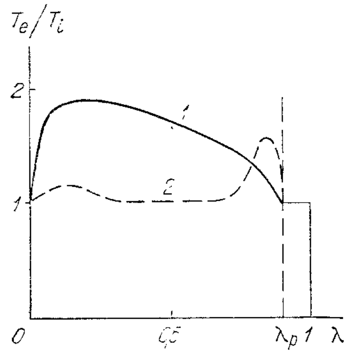

Fig. 1. Dependence of the temperature ratio T_e/T_i for different cases: 1) B < 0.1; 2) B > 0.1.

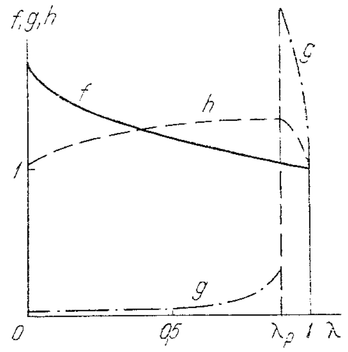

Fig. 2. Typical profiles when B < 0.1.

where all the functions are referred to their values behind the shock wave. Substituting the variables in Eq. (8) into Eq. (1), a system of differential (ordinary) equations is obtained and must be integrated in the interval $0 \leqslant \lambda \leqslant 1$. The mass and energy balance of the gas is written in the form [4]

$$\int_0^{\lambda_p} \frac{\delta(\gamma+1)}{\gamma-1} g d\lambda = \frac{m_0}{\rho_0 Q_0^{1/3}}, \quad \int_0^1 \frac{4\delta^2}{\gamma^2-1}(\theta+f^2) g d\lambda = \frac{q_0}{\rho_0 Q_0}. \tag{9}$$

For a source at the surface x = 0, it may be found, using Eq. (2), that

$$w_0 = q_0 \exp\left(-B\int_0^{\lambda_p} g^2/\theta_e^2 d\lambda\right), \quad B = \frac{\varkappa_0 \rho_0^2}{Q_0}\frac{(\gamma+1)^6}{4(\gamma-1)^2\delta^4}.$$

The boundary conditions at $\lambda = 0$ take the form

$$g(0) f(0) = \frac{m_0(\gamma-1)}{2\delta\rho_0 Q_0^{1/3}}, \quad \gamma\theta(0) + f^2(0) = \frac{(\gamma+1)^2}{2\delta^2}\frac{w_0}{m_0 Q_0^{2/3}},$$

and $f = g = h = \theta = 1$ at $\lambda = 1$. The mass of gas covered by the shock wave lies between the contact surface and the shock wave; therefore, the relation $\int_{\lambda_p}^1 g d\lambda = (\gamma-1)/(\gamma+1)$ is added to the integral in Eq. (9) and must be used to monitor the calculation.

The results of certain calculations are shown in Figs. 1 and 2. It is evident that the solution depends strongly on B. When B < 0.1, ionized gas absorbs the external radiation through its volume, and is heated, when the separation between the electron and ion temperature may reach $T_e/T_i = 1.5$–2. When B > 0.1, the main part of the incoming energy is absorbed by the contact surface, where nonequilibrium effects lead to the appearance of a "heating tongue"; the main mass of gas, however, has an equalized temperature. The other dimensionless parameters appearing in the problem have little influence on the qualitative picture of the flow. Taking account of particular physical processes in the general case leads to loss of self-similarity. The solutions obtained may serve as accurate data for the calculation of non-self-similar problems by finite-difference methods.

Note, in conclusion, that self-similar solutions also exist when the viscosity and heat conduction are taken into account [3-5], and also when $\rho_0 = \rho_\infty x^{-\omega}$ ($\omega < 1$, $p_0 = u_0 = 0$). In this case, assuming power-law dependences of the form $\mu = \mu_0 \rho^{\beta} \varepsilon_i^{\beta_1}$, $k = k_0 \rho^{\beta_2} \varepsilon_e^{\beta_3}$, μ_0, k_0 = const for the viscosity μ and thermal conductivity k, the following self-similarity conditions are obtained

$$\beta_1 = \frac{3 + 2n - \omega[n + 2 - \beta(n + 3)]}{2(n + \omega)},$$

$$\beta_3 = \frac{3 + 2n - \omega[n + 2 - \beta_2(n + 3)]}{2(n + \omega)},$$

$$\alpha_1 = \frac{\omega - 3}{2(n + \omega)}, \quad \alpha_2 = \frac{(n + 3)(\alpha_3\omega - 1)}{2(n + \omega)}.$$

NOTATION

p, pressure; T, temperature; ε, internal energy; ρ, density, u, velocity; γ, adiabatic index; x, t, spatial and time coordinates; Q, influx of energy from the external radiation to the gas particles; c_i, T_i, specific heat and temperature of ionic gas; c_e, T_e, specific heat and temperature of electron gas; p_i, p_e, partial pressures of ionic gas and electron gas; Q_{ei}, energy transfer between the electron and ion gases; $\varkappa$, coefficient of absorption of external radiation; q, intensity of external radiation; X, $\tilde{X}$, operators; λ, dimensionless coordinate; f, h, g, dimensionless functions; $\varkappa_0, q_0, Q_0$, constants.

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